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UTILITY THEORY*†

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Utility theory is interested in people's preferences or values and with assumptions about a person's preferences that enable them to be represented in numerically useful ways. The first two sections of this paper say more about what utility is, why people are interested in it, and how it is interpreted and used in the management and behavioral sciences. The third section summarizes a number of utility theories: it may be used either as a concluding overview of the range and variety of utility theories or as a bridge to the final sections. The final eight sections comprise a semi-technical survey of particular theories for readers interested in greater depth.

1. Introduction

On the practical level, utility theory is concerned with people's choices and decisions. It is concerned also with people's preferences and with judgments of preferability,[1] worth, value, goodness or any of a number of similar concepts.

The usual raw materials on which a utility theory (there are many) is based are an individual's preference-indifference relation $\precsim$, read "is not preferred to", and a set X of elements $x, y, z, \cdots$ usually interpreted as decision alternatives or courses of action. $\precsim$ is taken to be a binary relation on X, which simply says that if x and y are in X then exactly one of the following two statements is true:

1. $x \precsim y$ (x is not preferred to y)
2. not $x \precsim y$ (it is false that x is not preferred to y).

The relations of strict preference ($x < y$: y is preferred to x) and indifference ($x \sim y$: x is indifferent to y) are defined from $\precsim$ thus:

$$x < y \text{ means that } x \precsim y \text{ and not } y \precsim x$$
$$x \sim y \text{ means that } x \precsim y \text{ and } y \precsim x.$$

If not $x \precsim y$ and not $y \precsim x$, x and y are sometimes said to be incomparable.

A utility theory is essentially

1. a set of internally-consistent assumptions about X and the behavior of $\precsim$ on X

and

2. the theorems that can be deduced from the assumptions.

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[1] Some behavioral scientists insist that preferences are revealed *only* through choice-making behavior and not through verbal responses to questions of what is preferred. This viewpoint is not adopted in the present discussion.

An example of an assumption is the transitivity assumption

$$\text{if } x \precsim y \quad \text{and} \quad y \precsim z \quad \text{then} \quad x \precsim z$$

which appears in most utility theories.[2] An example of a theorem is:

> Numbers (numerical utilities) $u(x)$, $u(y)$, $\cdots$ can be assigned to the elements in X in such a way that, for all x and y in X,
>
> $$x \precsim y \quad \text{if and only if} \quad u(x) \leq u(y).$$

Many of the theorems, as is the case with our example, show how the assumed preference structure can be transformed into a corresponding numerical utility structure.

Most utility theories, when stripped of all nonmathematical interpretation, amount to abstract mathematical theories of binary relations. Their interest derives however not so much from their purely mathematical aspects as from what they say about preferences and decisions.

Despite agreement on what utility theory pertains to, there are a number of viewpoints on how it pertains. The different viewpoints arise from different interpretations of what the assumptions mean with respect to preferences and choices or decisions. These interpretations have grown up in the disciplines that are interested in the theory, primarily economics, psychology, statistics, and management science.

Interpretations of utility theory are often classified under two headings, prediction and prescription.[3] The predictive approach is interested in the ability of a theory to predict actual choice behavior. The prescriptive approach is interested in saying how a person ought to make a decision. Psychologists are primarily interested in prediction; economists in both prediction and prescription. In statistics the emphasis is on prescription in decision making under uncertainty. The emphasis in management science is prescriptive also.

This paper emphasizes a prescriptive interpretation of utility theory. The next section discusses ways that the theory may help decision makers, and comments on utility theory in psychology and consumer economics. Section 3 then summarizes various theories of utility. Ensuing sections go into these theories in greater detail.

Other surveys with their particular emphases are available. Edwards [82, 86], Luce and Suppes [183], and Becker and McClintock [33], oriented toward psychologists, provide extensive coverage and bibliographies. Arrow [15, 16], Majum-

[2] Armstrong [14] and Luce [176], for example, do not assume that indifference ($\sim$) is transitive. See also Armstrong [12, 13], Scott and Suppes [253], Suppes and Zinnes [280], Scott [252], and Krantz [165] on this point. See Tullock [292] for a defense of transitivity, and Michalos [198, 199] for criticism of transitivity. Papandreou [214], Davis [67], and Griswold and Luce [132] offer experimental evidence in support of transitivity. May's results [196] cast some doubt on transitivity with multidimensional alternatives. (In a related vein Shepard [256] obtains a complete breakdown of transitivity in judgments of relative pitch.)

[3] Marschak [193] uses "descriptive" and "normative", respectively.

dar [185] and Simon [259] discuss decision making from the prescriptive and predictive viewpoints. Houthakker [144] surveys utility theory (1935 to 1961) as it pertains to consumer economics. Parts of Edwards [82] and the other reviews also discuss utility theory in economics. Strotz [272], Alchian [5], and Marschak [191] give elementary introductions to expected utility theory, and Adams [2] presents a survey of this subject. Robertson [238, 239] swings through a number of topical areas of utility theory in an easy-going style. Davidson, McKinsey, and Suppes [63] give a philosophically-oriented introduction to utility, and Churchman [52] provides a philosophical analysis of decision and value theory, oriented towards operations research.

2. Interpretations of Utility Theory

The assumptions of a utility theory are usually stated in terms of an individual's preference-indifference relation $\preccurlyeq$ ("is not preferred to") applied to a set X of alternatives. Each assumption may be placed in one of the following three categories:[4]

1. Pure existential assumptions
2. Pure preference assumptions
3. Existential-preference assumptions.

A pure existential assumption refers to the structure of the decision problem and does not contain $\preccurlyeq$. "The number of alternatives is finite" is a pure existential assumption.

A pure preference assumption does not assume the existence of any particular elements in the situation at hand, and is often of the "if . . . then . . ." variety. The transitivity assumption "if x, y, and z are alternatives in X, and $x \preccurlyeq y$, and $y \preccurlyeq z$; then $x \preccurlyeq z$" and the connectivity assumption "if x and y are in X, then $x \preccurlyeq y$ or $y \preccurlyeq x$ (possibly both)" are pure preference assumptions.

Existential-preference assumptions mix existence conditions with preferences: "There exist alternatives x and y in X such that $x \prec y$ {i.e., $x \preccurlyeq y$ and not $y \preccurlyeq x$}." Another example is "if x and y are in X and $x \prec y$, then there is an alternative z in X such that $x \prec z$ and $z \prec y$."

From these examples we quickly realize that most of the assumptions serve a purpose of simplification or explication in that they give order and structure to an individual's preferences. The effect of this on utility structures will become apparent in following sections.

The categorization given above refers to the formal content but not to the behavioral interpretations of the assumptions. It is to interpretations and purposes that we now turn.

Prescriptive Utility Theory

In prescriptive utility theory a preference assumption is often viewed as a common-sense guideline for the individual to follow in identifying his preferences explicitly. It is a logic-like criterion of consistency and coherence, recommended

[4] A slightly different categorization is given by Suppes [277].

to the individual as a rule he ought to adopt in computing preferences. Interpreting transitivity in this manner we have: "If, for you, $x \prec y$ and $y \prec z$, then common sense strongly suggests that you should not prefer x to z. Hence, it is recommended that you adopt the transitivity rule as part of your decision-making policy."

Another variety of preference assumption found in some versions of prescriptive utility theory is a simplifying assumption that is not taken as a universal, common-sense guideline. If such an assumption is considered in a specific context, some effort should be made to test its credibility. One such assumption arising in what is called expected utility theory (Section 8) is: "If x and y are amounts of money that may accrue to the individual, and if p is a number between zero and one, then he is indifferent between receiving the amount

$$px + (1 - p)y$$

outright or taking a gamble from which he wins x with probability p or y with probability $1 - p$ (not both)." This has the effect of making the utility of each amount of money equal to the amount. Other examples of simplifying assumptions arise in multidimensional situations (Sections 5 and 9) and permit one to set the utility of a multidimensional alternative equal to the sum of utility numbers associated with each component of the alternative.

There are several interrelated purposes of prescriptive utility theory, all of which contribute to the decision-making process. We mention three here.

1. As already remarked, the theory serves as a normative guide in helping the decision maker codify his preferences. If the individual's preferences appear to violate a "rational" preference assumption, the theory suggests that he reexamine and revise one or more preference judgments to eliminate the inconsistency. It does not tell him which particular judgment(s) to revise.

2. The theory aims to help a decision maker "discover" or determine his preferences between complex alternatives. Among the reasons that make it difficult to make preference comparisons are multidimensionality and uncertainty. The former is characterized by cases in which the decision maker considers each alternative on the basis of many factors or attributes that can span a number of time periods. He may find it difficult to arrive at an overall preference between two alternatives when one is better than the other for some factors but the reverse is true for other factors. Characteristic of the uncertainty problem are cases in which the decision maker is uncertain of what will happen if he selects and implements an alternative or course of action. Many practical decision problems undoubtedly contain both these difficulties.

In trying to determine preferences between complex alternatives, the following procedure can be used. First, the individual makes some preference judgments that he feels fairly certain of. Some of these judgments may be between "simplified alternatives" that contain aspects of the actual alternatives, but are less complex than the actual alternatives. Using the utility theory, his preference data are transformed into corresponding numerical utility data. The numerical data are then manipulated in an attempt to compute or derive numerical utility

comparisons between the actual alternatives. The derived numerical comparisons are then transformed back into derived preference statements.

Partly because of the difficulty of comparing complex alternatives, several authors[5] have criticized the reasonableness of the connectivity assumption "if x and y are alternatives in X, then $x \preccurlyeq y$ or $y \preccurlyeq x$". In full recognition of the difficulty of comparing complex alternatives, some people still feel that, in principle, any two alternatives are comparable in preference. This does not assume that such comparisons are easy to make. Rather, from one prescriptive viewpoint, it presents the challenge of helping the decision maker "discover" his preferences between alternatives he finds difficult to compare.

3. A third purpose of prescriptive utility theory is to enable the decision maker's preferences to be transformed into a numerical utility structure to be used in an optimization algorithm. In many cases available alternatives are characterized by a set of involved statements or mathematical expressions. If the decision maker's utility structure has desirable mathematical properties, it may be possible, using appropriate techniques, to determine the available alternative with the greatest utility.[6]

Before going on to other views of how utility theory relates to behavior, we raise two questions. First: "Is utility theory actually used?" Although the use of utility theory involves several difficulties, including the actual measurement of utilities, its use has not been ignored. Marketing research has used parts of the theory in brand-preference and related research [131, 168, 189, 265], food industries have applied the theory in the quality control of their products [234], and additive utility theories (Section 5) have been used in the selection of corporate strategies [55, pp. 150–152], evaluation of product defects [271], assignment of electronic equipment to ships [24, 25, 281], and evaluation of tank power-system alternatives [94]. Indeed, innumerable persons (who may never have heard of utility theory) have probably "applied" the theory since at heart it purports to be the common sense of decision making. Anyone who has ever attempted to resolve the multidimensionality problem by weighing the pros and cons of the alternatives has used a version of additive utility theory. Anyone who has ever attempted to resolve the uncertainty problem by comparing his expectations as to what might result from the alternatives has used a version of expected utility theory. Proponents of prescriptive utility theory believe that familiarity with the formal theory and methods of measuring utilities can be a valuable supplement to the usual "intuitive" approaches to decision making.

The second question is: "Is prescriptive utility theory a substitute for so-called factual, objective analysis?" Most utility theorists do not deny the poten-

[5] Aumann [21, 22, 23], Davidson, Suppes, and Siegel [65, Chapter 4], and Churchman [53, Section 8.13]. See also Coombs [59, Chapter 13], Tversky [299], Kannai [152], and Fishburn [96, Section 6.3], and "Quasi Orders" in Section 4.

[6] If the decision maker knew precisely which alternatives were available and had no difficulty in deciding which of these he preferred most, then he would have no need for formal applications of utility theory in the situation at hand. Rapoport [232, 233] criticizes utility theory in this connection.

tial value of whatever one envisions as "factual, objective analysis." They do recognize, however, that regardless of the "facts" available to the decision maker, he must still try to decide which course of action is "best" (most preferred) in the situation at hand. In addition, much recent work in prescriptive utility theory is concerned with the utility of "factual" information relevant to making a decision, and it aims to aid the decision maker in deciding what information to seek out (if any) in order to be better informed (Section 10). This recognizes that preferences depend on past experience and present expectations. If more experience (including "factual" information) is obtained before a decision is made, the individual's preferences may change.[7] Because of this, appliers of utility theory should take care to insure that the preference data are up-to-date.

Utility Theory in Consumer Economics

Economics is the father of utility theory.[8] It has provided a rich body of theory that has been used, extended, and modified by investigators in all disciplines concerned with utility.

In consumer economics the preference assumptions are often taken to characterize the choices of a "rational man", faced with deciding how much he ought to spend on various commodities. It is often assumed that, with or without prior deliberation, he will purchase the most preferred available "commodity bundle". This is really a prediction as to how he will behave. Assuming that the "rational man" acts according to the theory under consideration (a number of theories have been proposed), the effect of changes in commodity prices and income on his behavior is then investigated.[9] This also involves a prediction: if prices and income change in a certain way, the theory predicts that his choice will change in a certain way.

Extension of the individual utility-theory ideas to the economics of a society (set of consumers) has interested many economists. For example, there have been efforts to use the hypothesized behavior of individuals to predict the reaction of a social group to changes in commodity prices and incomes. Whether predicted effects of changes are based on utility theory or not, many economists are interested in evaluating these changes for social good or ill on the basis of the utilities of the individuals involved. Their interest here is in prescribing economic policies that will, in some sense, be beneficial to a society. The prescriptive use of utility theory in this context is classified under "welfare economics." (See, for example, Little [173] and Rothenberg [240].)

[7] Insofar as preferences are rooted to particular instances of time at which they are determined, it is wrong to say that preferences change. What changes is a person's experience. Because of this his "time t_1 preferences" may differ from his "time t_2 preferences".

[8] Stigler [270] gives a history of utility primarily from 1776 to 1915, and Kauder [153] traces the history of marginal utility theory.

[9] See, for example, Debreu [70], Uzawa [300], Hicks [141], Allen [7, Chapter 19], or Samuelson [242].

We shall say more about the determination of social policies or group decisions on the basis of individual utilities in Section 11.

Utility Theory in Psychology

Motivated in large part by the theoretical investigations of economists, psychologists (and others) have become interested in testing the predictive ability of several utility theories. Real efforts along this line began around 1950.

Psychologists are interested in describing or predicting actual choice behavior, whether or not it is "rational." The preference assumptions of a utility theory when viewed in this light are predictions. For example, the transitivity assumption states a prediction: "If a person prefers x to y and y to z, then (it is predicted that) he will prefer x to z."

The many experiments undertaken by psychologists in carefully designed repetitive-choice situations have indicated that most of the theories tested are not notoriously good predictors of choice behavior for many of the individuals tested. (See Edwards [82, 86], Luce and Suppes [183], and Becker and McClintock [33].) Although this may be discouraging to some, it has had the general effect of stimulating new developments. Dissatisfaction with earlier theories (pre 1957) has led to a new body of theory popularly called "stochastic utility theory" in which the assumptions are stated in terms of probabilities of choice rather than preferences. In the theories of Debreu [69, 72] and Suppes [278] the assumptions, stated in terms of binary choice probabilities $p(x, y)$—the probability that x will be chosen when the subject must choose between x and y—imply that numbers $u(x)$, $u(y)$, $\cdots$ can be assigned to x, y, $\cdots$ so that $p(x, y) \geqq p(z, w)$ if and only if $u(x) - u(y) \geqq u(z) - u(w)$. The books by Luce [178] and Restle [236] and the surveys by Becker, DeGroot, and Marschak [29] and by Luce and Suppes [183] provide good coverage of the topic.[10] Although the future may see more interest in prescriptive use of stochastic utility theory, its employment to date has been almost exclusively predictive.

Because the results of an individual's decision will often be influenced by the actions of others, prescriptive utility theory is interested in prediction. If it were possible to predict accurately the actions of other people (for example, customers or competitors), then the individual in the prescriptive theory would be that much better off. Decision makers do, of course, use a variety of methods to make such predictions, but most of these are not directly related to predictive utility theory, although a relationship between methods and theory can often be imputed.

At the present time, however, the predictive work in utility theory carried out by psychologists seems to offer little aid to the individual decision maker. As said above, most of the theories tested are not notoriously good predictors of choice behavior: even if they were, their relevance to the kinds of problems faced

[10] Some other contributors are Quandt [225], Luce [177], Georgescu-Roegen [118], Davidson and Marschak [66], Block and Marschak [42], Marschak [192], Audley [20], Chipman [51], Griswold and Luce [132], and Becker, DeGroot, and Marschak [30, 31, 32].

by managers would be questionable. For one thing, most practical situations differ from experimental ones in being more complex and nonrepetitive. In addition, the data required to predict others' choices by many of the proposed theories is quite sparse in practical situations.

One way that some discussants of prescriptive theory propose to use whatever information is available on other people whose actions affect the decision makers' actions, is to transform the available information into so-called subjective probabilities. These probabilities represent the decision maker's (or his aides') beliefs about what other people *might* do. As more information is obtained, the probabilities are revised and up-dated.[11] A prime example of this is found in Ward Edwards' Probabilistic Information Processing (PIP) Systems.[12] We shall say more about subjective probabilities in Section 10.

3. Preview and Summary of Theories

Having considered interpretations of utility theories, we now preview some of the theories discussed in more detail in ensuing sections. As final introduction to the sectional summaries, we remark that the pure preference and existential-preference assumptions used in the theories are of three main types:

1. *Order* These provide $\preccurlyeq$ with ordering properties such as connectivity and transitivity. (Assumptions 4.1, 8.1.)

2. *Archimedean* When the set of alternatives X is infinite, these ensure the existence of numerical utilities. (Assumptions 4.2, 8.3.)

3. *Independence* These serve a variety of purposes in giving utilities special properties beyond those obtained from order and Archimedean assumptions. (Assumptions 5.1_m, 5.1, 8.2 and 9.1.)

Besides the numbered assumptions, other forms of order, Archimedean, and independence assumptions are alluded to at various points. Several other special-purpose assumptions are used in some theories (for example, Assumptions 7.1 and 7.2).

Sectional Summaries

4. *Preference Orders and Utility Functions* When X has a finite number of alternatives and $\preccurlyeq$ completely orders the alternatives (according to preference), numerical utilities $u(x)$, $u(y)$, $\cdots$ can be assigned to the alternatives $x, y, \cdots$ in X so that $x \preccurlyeq y$ if and only if $u(x)$ is not greater than $u(y)$. When X is infinite, an Archimedean assumption may be needed to ensure that numerical utilities that mirror the preference order can be assigned to the alternatives. Several ramifications of this basic theory are noted.

5. *Utilities and Multidimensional Alternatives* When each of the alternatives $x, y, \cdots$ has a number of components, or when preferences between alternatives

[11] As with preferences, subjective probabilities are time-oriented, and it is incorrect to say that subjective probabilities change over time. What changes is a person's experience, and hence his "time t_1 probabilities" may differ from his "time t_2 probabilities".

[12] See Edwards [87] and, for additional discussion and references, Edwards and Phillips [89].

depend on many factors, an assumption of independence among components or factors implies that each of the numerical utilities $u(x)$, $u(y)$, $\cdots$ can be written as the sum of utilities assigned to the several components. With two-dimensional alternatives, if $x = (x_1, x_2)$, then $u(x) = u_1(x_1) + u_2(x_2)$. If there is a rigid priority preference among the underlying factors (factor 1 is overwhelmingly more important than factor 2), there might be no way to assign numerical utilities as in Section 4, but it may be possible to express the utility of $x = (x_1, x_2)$ as a vector of numbers $U(x) = (u_1(x_1), u_2(x_2))$ so that $(x_1, x_2) \preccurlyeq (y_1, y_2)$ if and only if $[u_1(x_1) < u_1(y_1)]$ or $[u_1(x_1) = u_1(y_1)$ and $u_2(x_2) \leqq u_2(y_2)]$.

6. *Time Preferences* When the multidimensionality of Section 5 refers to a succession of time periods and the same set of consequences is assigned to each time period (such as net income in each period), a number of new concepts arise, including impatience (discounting the future), extreme impatience (only immediate pleasures are valued), eventual impatience (discounting at some point in the future), time perspective (related to discounting), no time preferences (a gain tomorrow that is equal to a gain today has the same utility as measured today as does today's gain), persistence (if I prefer steak to chicken today I also prefer steak to chicken tomorrow), and variety (if I order steak today, I'll order chicken tomorrow). The problem of what our preferences will be in the future (instead of today) for future events is noted along with strategies for future planning.

7. *Utility Differences and Even-Chance Alternatives* With the assignment to the alternatives of utilities $u(x)$, $u(y)$, $\cdots$ that mirror the preference order $\preccurlyeq$ on X, suppose $u(x) < u(y)$ and $u(z) < u(w)$. If we envision that the difference in preference between x and y exceeds the difference in preference between z and w, we are tempted to write $u(y) - u(x) > u(w) - u(z)$. Assumptions implying that this can be done are identified. In the even-chance context, suppose you prefer a 50-50 gamble resulting in either \$0 or \$100 to a 50-50 gamble resulting in either \$10 or \$50. We are then tempted to write $\frac{1}{2}u(\$0) + \frac{1}{2}u(\$100) > \frac{1}{2}u(\$10) + \frac{1}{2}u(\$50)$ or $u(\$0) + u(\$100) > u(\$10) + u(\$50)$ or $u(\$100) - u(\$50) > u(\$10) - u(\$0)$. Assumptions implying that your utilities can be represented in this way are similar to those used in the preference-difference approach.

8. *Expected Utilities* Preferences between probability distributions (gambles) that employ all probabilities between 0 and 1 can, under the assumptions of Section 4, be represented by numerical utilities assigned to the distributions, with $u(x) = u(P)$ when P is a probability distribution that assigns probability 1 to the consequence x. An additional independence assumption implies that the utility of a distribution can be written as a weighted sum of the utilities of the consequences in X, the weights being the probabilities assigned by the distribution to the consequences. Comments on the expected utility of money introduce several simplifying assumptions into the discussion.

9. *Expectations and Multidimensional Consequences* When utilities can be assigned to probability distributions as in Section 8, and the basic consequences x, y, $\cdots$ are multidimensional as in Section 5, the utility of x can be written as

the sum of utilities assigned to its components under an additional, simplifying independence assumption. The notion of the overwhelming importance of some factors over others is also looked at in the expected-utility context. Many of the time-preference concepts of Section 6 apply to the theory of this section.

10. *Expected Utility and Subjective Probability* We envision alternatives whose resulting consequences depend on uncertain aspects of the environment. Assumptions on preferences between such alternatives lead to an assignment of utilities to the consequences and to the alternatives plus an assignment of subjective probabilities to the possible states of the environment in such a way that the utility of an alternative can be written as a weighted sum of the utilities of the consequences, the weight for any alternative-consequence pair being the subjective probability associated with the states of the environment that yield the given consequence when the given alternative is used. The question of which experiment should be performed from a set of experiments that can yield additional information on which state of the environment is, in fact, the true state is examined.

11. *Social Choice and Individuals' Preferences* "How shall a society's choice or a group's decision depend on the preferences of the individuals involved?" has received many answers, none of which is generally considered universally satisfying. Methods of majority rule and rating, both with long histories, are mentioned. Objections to these and related procedures are raised. A set of four conditions for social choice on the basis of individuals' preferences are noted to be logically incompatible.

4. Preference Orders and Utility Functions

Proposition 4.1 (Ordered Utilities). *A number $u(x)$ can be assigned to each x in X so that if x and y are in X, then*

$$x \preccurlyeq y \quad \text{if and only if} \quad u(x) \leqq u(y). \tag{4.1}$$

If this is true then the utility function u on X preserves the ordering of $\preccurlyeq$ and (4.1) permits us to go back and forth between preferences and utilities. What must be assumed about $\preccurlyeq$ in order for Proposition 4.1 to be true?

The "if and only if" of (4.1) requires that $\preccurlyeq$ on X be *connected* or complete [$x \preccurlyeq y$ or $y \preccurlyeq x$ (possibly both)] and transitive [if $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$].[13] These pure preference assumptions define a weak order (sometimes called a complete preorder).

Assumption 4.1 (Weak Order). *$\preccurlyeq$ is a weak order on X. That is, $\preccurlyeq$ on X is connected and transitive.*

Although necessary for the validity of (4.1), this assumption is not sufficient in all cases. The problem concerns the size of X. If Assumption 4.1 holds, the indifference relation $\sim$ on X [$x \sim y$ means that $x \preccurlyeq y$ and $y \preccurlyeq x$] is an *equivalence relation*: that is, $\sim$ is *reflexive* ($x \sim x$), *symmetric* ($x \sim y$ implies $y \sim x$),

[13] Interesting mathematical analyses of transitivity and connectivity are included in the papers by Eilenberg [90], Rader [226], Sonnenschein [263], and Uzawa [300].

and *transitive* ($x \sim y$ and $y \sim z$ imply $x \sim z$). The relation $\sim$ can then be used to partition X into a set of nonoverlapping subsets called *equivalence classes.* x and y are in the same equivalence class if and only if $x \sim y$.

If the number of equivalence classes under $\sim$ *is finite or denumerable,* then Assumption 4.1 implies Proposition 4.1 (see Suppes and Zinnes [280, pp. 26–28] for a proof). However, if the number of equivalence classes under $\sim$ is uncountable, then Assumption 4.1 is not sufficient for Proposition 4.1. For this case we use an additional assumption, with the definition that a subset Y of X is $\preccurlyeq$ *dense in* X if and only if for each pair x, z in X such that $x \prec z$ there is a y in Y such that $x \preccurlyeq y$ and $y \preccurlyeq z$.

Assumption 4.2 (Denseness). *There is a countable (finite or denumerable) subset of* X *that is* $\preccurlyeq$ *dense in* X.

Proposition 4.1 *is true if and only if Assumptions* 4.1 *and* 4.2 *are true*[14] (see Milgram [200, p. 27], Birkhoff [38, p. 32] or Luce and Suppes [183, pp. 263–264] for proof).

Let X be the set of all ordered pairs of numbers (x_1, x_2), and define $(x_1, x_2) \preccurlyeq (y_1, y_2)$ if and only if $[x_1 < y_1]$ or $[x_1 = y_1$ and $x_2 \leqq y_2]$, so that the first component of the pairs is dominating. Then $\preccurlyeq$ is a weak order on X, but Assumption 4.2 is false, and it is impossible to assign a number $u(x_1, x_2)$ to each pair (x_1, x_2) in X so that (4.1) holds for all x, y in X.[15]

Uniqueness Properties

If u on X is one numerical function satisfying (4.1), then another function v on X satisfies (4.1) if and only if $u(x) \leqq u(y)$ implies $v(x) \leqq v(y)$ and vice versa. This is abbreviated by saying that "u is unique up to order-preserving transformations". If additional conditions beyond (4.1) are desired for u, then its uniqueness properties may differ.

Simplifying Assumptions

Practical considerations lead to simplifying assumptions that permit us to specify conditions on u in addition to (4.1). One such condition is that u be continuous, which very roughly says that for any x in X there are other elements in X whose utilities are arbitrarily close to $u(x)$. General discussions of continuity use notions beyond the scope of this paper.[16] Simplifying assumptions relating to the problem of multidimensionality will be discussed in the next section.

Quasi Orders

$\preccurlyeq$ on X is a *quasi order* if it is reflexive ($x \preccurlyeq x$) and transitive. This implies that $\sim$ is an equivalence relation as when $\preccurlyeq$ is a weak order. If $\preccurlyeq$ is a quasi

[14] Theorem 7, p. 28 in Suppes and Zinnes [280] is identical to this. Chipman [50, pp. 210–211] criticizes Assumption 4.2: his Theorem 3.3 (p. 214) gives alternative assumptions that imply Proposition 4.1. See also Debreu [68], Newman and Read [212], and Murakami [207].

[15] See, for example, Debreu [68] or Luce and Suppes [183, p. 261].

[16] Luce and Suppes [183, pp. 264–267], Debreu [68, 70, 73], Newman and Read [212], Eilenberg [90], Rader [226], Wold [310], Wold and Jureen [312], Yokoyama [314, 315], Uzawa [300], and Lloyd, Rohr, and Walker [175] discuss continuity.

order, then, for any x, y in X, either $x < y, y < x, x \sim y$, or x and y are not comparable or connected, meaning that neither $x \preccurlyeq y$ nor $y \preccurlyeq x$ holds. Proposition 4.1 cannot hold for a quasi order unless it is connected, whereupon it becomes a weak order. Because of this, (4.1) in a quasi-ordered preference structure is amended to read: $x < y$ implies $u(x) < u(y)$, and $x \sim y$ implies $u(x) = u(y)$. $u(x) < (=)u(y)$ does not imply $x < (\sim)y$ unless $\preccurlyeq$ is connected.

Interest in quasi orders has stemmed either from dissatisfaction with connectivity or a desire to generalize the theory (or both). They are discussed by Davidson, Suppes, and Siegel [65, Chapter 4], Aumann [21, 22], Fishburn [96, Sections 4.4 and 6.3], Kannai [152], Aumann [23], and Tversky [299]. The first four and the last include considerations in expected utility theory (Section 8). Kannai's paper discusses lexicographic utilities, and the last two discuss additive utilities (Section 5). Szpilrajn [283] proves a theorem that is equivalent to the assertion that for any given quasi order there is a weak order that is consistent with the quasi order. Arrow [18, Chapters IV and VI] extends Szpilrajn's result and discusses quasi orders in the social-choice context (Section 11).

5. Utilities and Multidimensional Alternatives

Two closely-related forms of multidimensional preferences that lead to additive utilities and lexicographic utilities are examined in this section. Some additive structures are compensatory:[17] they deal with how much the utility of one factor must be increased to offset a decrease in the utility of another factor. Lexicographic utility structures include cases where some factors are "overwhelmingly more important" than others.

Definitions

For ease in exposition we assume that there are n factors of concern to the decision maker, denoted as $X_1, X_2, \cdots, X_n$. Each X_i is a set. An element x_i in X_i is a "level of the factor X_i". In an allocation problem x_i may be the resources allocated to the i^{th} activity, $i = 1, 2, \cdots, n$. Then X_i is the set of all possible allocations to the i^{th} activity.

The Cartesian product of the X_i, denoted $X_1 \times X_2 \times \cdots \times X_n$, is the set of all n-tuples $(x_1, x_2, \cdots, x_n)$ with x_i in X_i for each i. x_i is the i^{th} component of $(x_1, x_2, \cdots, x_n)$. In the allocation example each $(x_1, x_2, \cdots, x_n)$ is a complete allocation over the n activities. *In this section X is taken to be equal to or a subset of $X_1 \times X_2 \times \cdots \times X_n$.* Additivity is concerned with the following proposition.

Proposition 5.1 (Additive Utilities). *A number $u_i(x_i)$ can be assigned to each x_i in X_i, for $i = 1, 2, \cdots, n$, so that if $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ are in X then*

[17] Coombs [59, Section 9.2 and Chapter 13]. Additivity and related concepts are discussed in the consumer-demand context by Fisher [109], Leontief [171], Samuelson [242, 244], Houthakker [143, 145], Strotz [274, 275], Rajaoja [229], Frisch [116], Koopmans [160], Sono [264], Gorman [129], Tolley and Giesman [289], and Barten [28], among others.

(5.1) $x \preccurlyeq y$ *if and only if* $u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n)$

$$\leq u_1(y_1) + u_2(y_2) + \cdots + u_n(y_n).$$

The lexicographic-utility proposition requires the following definition. If $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ are n-dimensional vectors of numbers, then $(a_1, a_2, \cdots, a_n) \overset{L}{\leq} (b_1, b_2, \cdots, b_n)$ if and only if $a_1 < b_1$ or $[a_1 = b_1$ and $a_2 < b_2]$ or $[a_1 = b_1, a_2 = b_2$, and $a_3 < b_3]$ or $\cdots$ or $[a_i = b_i$ for all $i < n - 1$ and $a_{n-1} < b_{n-1}]$ or $[a_i = b_i$ for all $i < n$ and $a_n \leq b_n]$. $\overset{L}{\leq}$ is called a *lexicographic order*. A lexicographic order on a set of n-dimensional vectors of numbers is a weak order. In the 2-dimensional example following Assumption 4.2, $\preccurlyeq$ is a lexicographic preference order.

Proposition 5.2 (Lexicographic Utilities). *A number $u_i(x_i)$ can be assigned to each x_i in X_i, for $i = 1, 2, \cdots, n$, so that if $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ are in X then*

(5.2) $x \preccurlyeq y$ *if and only if* $(u_1(x_1), u_2(x_2), \cdots, u_n(x_n))$

$$\overset{L}{\leq} (u_1(y_1), u_2(y_2), \cdots, u_n(y_n)).$$

Letting $u(x) = u_1(x_1) + \cdots + u_n(x_n)$, (5.1) says that (4.1) holds, and the utility of a vector equals the sum of utilities of its components. Instead of single-valued utilities, (5.2) works with utility vectors $U(x) = (u_1(x_1), \cdots, u_n(x_n))$.

Examples and Applications

Applications of additive utilities have been reported in [24, 25, 55, 80, 94, 127, 260, 262, 271, 281, 313]. These papers also discuss methods of measuring additive utilities.[18] Some applications involve the evaluation of subsets of a given set of n elements:[19] each X_i has two elements—$x_i = 1$ if the i^{th} element is in the subset and $x_i = 0$ otherwise. Each n-dimensional vector of 0's and 1's represents a subset. If $u_i(0) = 0$ in (5.1), then the utility of a subset equals the sum of utilities of the elements in the subset. The n elements may be research proposals [127, 262], objectives [54, 55], criteria [80], product defects [271] and so forth. If the decision problem is to select a maximum-utility subset, subject to linear constraints on the x_i, the algorithms of Balas [26] and Glover [121] can be used.[20]

Other applications require larger X_i sets. A variety of allocation problems enters here, where x_i is the quantity of a resource or resource vector to be allocated to a certain activity [24, 25, 260, 281, 313]. Or X_i may be the set of possible levels of the i^{th} performance criterion for a system under development [94]. Optimization algorithms for such problems involve a large part of the extensive literature of mathematical programming.

Applications of lexicographic utilities appear limited because of the "over-

[18] See also [54, 96, 98, 99, 124, 184, 197, 205, 219, 224, 284, 287, 288, 290]. Most of the estimation methods are summarized in [105].

[19] Credit for the additivity theory for this case is due to Kraft, Pratt, and Seidenberg [163].

[20] Freeman [111] discusses computational experience.

whelming importance" notion. For an example, if a prisoner of war is unwilling to reveal vital information regardless of the torture he may undergo, his utilities could be represented as in (5.2). Prisoners tortured to death for withholding information from the enemy bear grim witness to the reality of this example. Other examples are given by Chipman [50, p. 221], Georgescu-Roegen [118, pp. 518–520], Newman and Read [212, p. 152], and Coombs [59, p. 201].

Independence Assumptions

The crucial assumptions for (5.1) and (5.2) are independence assumptions based on equivalence relations T_m on $X \times X \times \cdots \times X$ (m times) for $m = 2, 3, \cdots$ *ad infinitum*.

Definition 5.1. $(x^1, x^2, \cdots, x^m)T_m(y^1, y^2, \cdots, y^m)$ *if and only if* $m \geq 2$ *and*

(1) $x^1, \cdots, x^m, y^1, \cdots, y^m$ *are all in* X;

(2) *with* $x^j = (x_1^j, x_2^j, \cdots, x_n^j)$ *and* $y^j = (y_1^j, y_2^j, \cdots, y_n^j)$ *for* $j = 1, 2, \cdots, m$, $(y_i^1, y_i^2, \cdots, y_i^m)$ *is a permutation* (*reordering*) *of* $(x_i^1, x_i^2, \cdots, x_i^m)$ *for each* i *from* 1 *to* n.

The following illustrate T_3 when $n = 4$:

$$\begin{array}{ll} x^1 = (1, a, .03, \$10) & y^1 = (2, c, .03, \$50) \\ x^2 = (2, b, .05, \$50) & y^2 = (1, b, .08, \$50) \\ x^3 = (3, c, .08, \$50) & y^3 = (3, a, .05, \$10). \end{array}$$

If the x^j and y^j are in X, then $(x^1, x^2, x^3)T_3(y^1, y^2, y^3)$ since (2, 1, 3) is a permutation of (1, 2, 3), (c, b, a) is a permutation of (a, b, c), and so on. If (5.1) holds and $(x^1, \cdots, x^m)T_m(y^1, \cdots, y^m)$, then $u(x^1) + u(x^2) + \cdots + u(x^m) = u(y^1) + u(y^2) + \cdots + u(y^m)$, so that if $u(x^j) \leq u(y^j)$ for all $j < m$, then $u(y^m) \leq u(x^m)$. This is reflected directly in the following assumptions.

Assumption 5.1$_m$ (Independence). *If* $(x^1, x^2, \cdots, x^m)T_m(y^1, y^2, \cdots, y^m)$ *and* $x^j \lesssim y^j$ *for* $j = 1, 2, \cdots, m - 1$, *then* $y^m \lesssim x^m$.

Assumption 5.1. Assumption 5.1$_m$ *holds for* $m = 2, 3, \cdots$, *ad infinitum*.

If $\lesssim$ is reflexive ($x \lesssim x$), Assumption 5.1$_m$ implies Assumption 5.1$_r$ for $r < m$. If 5.1$_3$ holds, then $\lesssim$ is transitive, for $(x, y, z)T_3\,(y, z, x)$; therefore, if $x \lesssim y$ and $y \lesssim z$, then $x \lesssim z$.

More to the point, *Assumption* 5.1 *is true if Proposition* 5.1 *is true* [see above] *or if Proposition* 5.2 *is true* [from the fact that $u_i(x_i^1) + \cdots + u_i(x_i^m) = u_i(y_i^1) + \cdots + u_i(y_i^m)$ for each i].

Additive Utilities

Building on the pioneering work of Adams and Fagot [4] and Scott and Suppes [253], Adams [3], Scott [252], and Tversky [295] have shown that *if* X *is a finite set equal to or a subset of* $X_1 \times X_2 \times \cdots \times X_n$, *then Proposition* 5.1 *holds if and only if* $\lesssim$ *on* X *is connected and Assumption* 5.1 *holds*. Scott and Suppes [253] show that this statement is false if Assumption 5.1 is replaced by 5.1$_m$ for any finite m. Tversky [299] shows that it is true even if X is not finite, provided

that an appropriate Archimedean assumption holds (to insure single-valued utilities).

By using some existential assumptions, several authors obtain (5.1) with only part of Assumption 5.1. With $n = 2$, Debreu [72] has shown that a consequence[21] of Assumption 5.1_3 and several other assumptions (weak order, a pure existential assumption, and an Archimedean-continuity assumption) imply Proposition 5.1. For $n \geq 3$ Debreu proved that, under his other assumptions, 5.1_3 may be replaced by 5.1_2 provided that at least thr.e factors actively influence preferences. For $n = 2$ Luce and Tukey [184] use a consequence[21] of 5.1_3, weak order, and existential and Archimedean assumptions different than Debreu's to imply Proposition 5.1. Luce [179] extends the Luce-Tukey theory to $n \geq 3$ using Assumption 5.1_3. Krantz's theory [164] for the n-dimensional case uses a consequence of Assumption 5.1_{n+1}. Each theory in this paragraph assumes that $X = X_1 \times X_2 \times \cdots \times X_n$, requires each X_i to be infinite[22], and implies that functions v_i on X_i satisfy (5.1) in place of the u_i *if and only if* there are numbers $b_1, b_2, \cdots, b_n$ and $a > 0$ such that

$$v_i(x_i) = au_i(x_i) + b_i \quad \text{for each} \quad x_i \text{ in } X_i,\ i = 1, 2, \cdots, n. \tag{5.3}$$

Thus, if we wish u on X to satisfy

$$u(x_1, x_2, \cdots, x_n) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n) \tag{5.4}$$

as well as (4.1), then by (5.3) u is "unique up to *linear*, order-preserving transformations" of the form[23]

$$v(x) = au(x) + b\ [b = b_1 + b_2 + \cdots + b_n \text{ in } (5.3)]. \tag{5.5}$$

If the X_i are finite sets, the u_i can be transformed as in (5.3) with (5.1) still valid, but (5.3) *does not* exhaust all transformations of the u_i that preserve the order of $\overset{i}{\prec}$ in (5.1). In any event *if* $X = X_1 \times X_2 \times \cdots \times X_n$ and (5.1) holds, then there is a unique weak order $\overset{i}{\prec}$ on X_i defined by $x_i \overset{i}{\prec} y_i$ if and only if $u_i(x_i) \leq u_i(y_i)$.[24] This is not universally true when X is a subset of $X_1 \times X_2 \times \cdots \times X_n$.

Lexicographic Utilities

Connectivity of $\prec$ and Assumption 5.1 are necessary for Proposition 5.2 but not sufficient since they say nothing about the domination of X_1 over X_2,

[21] If $(x_1, x_2) \prec (y_1, y_2)$ and $(y_1, z_2) \prec (z_1, x_2)$, then $(x_1, z_2) \prec (z_1, y_2)$. This is called a cancellation assumption. The independence assumption 5.1 may be viewed as all possible cancellation assumptions in the additive context. Krantz [164, 166] gives an alternate proof of the Luce-Tukey theorem.

[22] Except for trivial cases.

[23] That is, under these theories, there exist functions u satisfying (4.1) that also satisfy (5.4), and all such functions are related as in (5.5). Stevens [266] introduced the term *interval* measure for functions unique up to linear, order-preserving transformations.

[24] Certain difference comparisons must also remain invariant under transformations that preserve order.

X_2 over $X_3, \cdots$ as implied by the lexicographic order. If X is finite, these two conditions are sufficient for Proposition 5.1; so that Proposition 5.2 implies Proposition 5.1 for finite X. Hence, if (5.2) holds when X is finite, we have our choice of working with lexicographic or additive utilities. The former would often be easier to work with.

Both (5.1) and (5.2) can hold when X is infinite. Let X be the denumerable set of all pairs (i, j) of nonnegative integers with $(i, j) \precsim (k, m)$ if and only if $(i, j) \overset{L}{\leq} (k, m)$. Then $u_1(i) = i$ and $u_2(j) = j/(1 + j)$ satisfy (5.1) *and* (5.2). However, $u_1(i) = i$ and $u_2(j) = j$ satisfy (5.2) but not (5.1). In a similar way $u_1(i) = i$ and $u_2(j) = j/(1 + j)$ satisfy both (5.1) and (5.2) when X is the *uncountable* set of all pairs (i, j) in which i is any nonnegative integer and j is any real number greater than or equal to zero. (Newman and Read [212, pp. 157–158] give some general comments on this point.)

Thus, lexicographic utilities are not necessarily incompatible with numerical utilities, a point stressed by Chipman [50] and Newman and Read [212]. Indeed, X need not have a product structure to define lexicographic utilities. Let X have three elements, a, b, c with $a \prec b \prec c$. Defining three-dimensional utility vectors $U(a) = (0, 0, 1)$, $U(b) = (0, 1, 0)$ and $U(c) = (1, 0, 0)$, we get $x \precsim y$ if and only if $U(x) \overset{L}{\leq} U(y)$. Chipman's Theorem 3.1 generalizes this idea.

Lexicographic utilities are discussed also by Georgescu-Roegen [118], Hausner [139], Thrall [286], Kannai [152], and Fishburn [106].

6. Time Preferences

At the present time a person may contemplate a sequence of decisions to be made or a sequence of events to be experienced during the future. Let X be the set of all conceivable sequences, one of which will be experienced as time passes. Then the theory of Section 4 applies to X as does the theory of Section 5 when X_i is taken to be the set of events from which one will be experienced during period i from now. If (5.1) holds, the utility of any n-tuple of events $(x_1, x_2, \cdots, x_n)$ in X, as viewed from the present, equals the sum of utilities assigned to each event. u_i in (5.1) is the person's *present* utility function for the i^{th} period hence. His utilities for a fixed calendar period may differ at different times prior to the period.

Many investigations of time preferences assume a *homogeneous* structure for X, so that the set of events in each period is the same. Call the event set A. Then the i^{th} component in the n-tuple $(x_1, x_2, \cdots, x_n)$ is the event in A experienced in the i^{th} period hence. If (5.1) holds, the u_i, all defined on A, may differ for different i, especially u_n, since it would presumably include the person's present expectations of the future beyond period n when x_n actively influences that future. Because of time-horizon problems some authors prefer to work with denumerable vectors $(x_1, x_2, \cdots$ ad infinitum).[25]

In brief fashion we now review seven notions of preferences in the time context. For simplicity, assume a homogeneous structure for X.

[25] Fishburn [102] states necessary and sufficient conditions in terms of u for additivity in this case.

1. *Impatience* (discounting the future). If in a given period event a is preferred to b, then the sequence $(x_1, \cdots, a, \cdots, b, \cdots)$ is preferred to $(x_1, \cdots, b, \cdots, a, \cdots)$ where the sequences are the same except that a and b are interchanged. This has been investigated in the denumerable-period context by Koopmans [160] and Koopmans, Diamond, and Williamson [162]; Diamond, Koopmans, and Williamson [78] gives a summary. For discounting in the continuous-time context see, for example, Strotz [273]. Feldstein [95] and Marglin [187] discuss social time preference discount rates.

2. *Extreme Impatience*. Lexicographic utilities as in (5.2) could apply to persons overwhelmingly concerned with immediate pleasures.

3. *Eventual Impatience*. Discounting might not occur in the near future, but all periods sufficiently far away will be discounted. This is discussed by Diamond [77] in the denumerable-period context.

4. *Time Perspective*.There exists a utility function satisfying (4.1) such that as the timing of relevant differences between event sequences recedes into the future, the utility difference between the sequences diminishes. (See Koopmans, *et al.* [162].)

5. *No Time Preferences*. At the present time the individual neither discounts nor "overcounts" the future with respect to the present. For example, if (5.1) holds and $(x_1, x_2, \cdots, x_n) \sim (y_1, y_2, \cdots, y_n)$ when one n-tuple is a permutation of the other, then the u_i (on A) may be made equal to one another. This is commented on by Wold and others [311], looked at in the finite-period context by Fishburn [100], in the denumerable-period context by Diamond [77], and in the continuous-time context by Ramsey [230].

6. *Persistence* (or stationarity). For example, if a is preferred to b in one period it will be preferred to b in any other period. Persistence is used by Koopmans [160], Koopmans, *et al.* [162], and Diamond [77].

7. *Variety* (nonpersistence). For $n = 2$ let x_1, x_2 be respectively the meats a person has for dinner tonight and tomorrow night. Suppose (steak, steak) $<$ (steak, lobster) and (lobster, lobster) $<$ (lobster, steak). This violates[26] Assumption 5.1_2; note particularly the second components in these pairs. In a word, this person likes variety in his diet. (Again, see Wold and others [311].)

All these notions apply to preferences concerning the future, *viewed from the present*. In making a decision we are often uncertain about what our preferences for future events will be *in the future*. If the problem is to select an event now (the first component of a sequence), we do not have to precommit ourselves to later choices. *However*, our present choice may limit (or expand) what we can select in the future. Koopmans [161], concerned with the problem of present choice and future freedom of choice, discusses several assumptions that apply $\lessdot$ not to sequences of choices (or events) in X, but rather to subsets of X, in any one of which all sequences have the same initial component. In a slightly different setting Klein and Meckling [157] suggest that in problems beset by future uncertainties the best strategy may be to concentrate attention on immediate de-

[26] Since ((steak, steak), (lobster, lobster)) T_2 ((steak, lobster), (lobster, steak)) and $(s, s) < (s, l)$ and $(l, l) < (l, s)$.

cisions that lead toward the main objective while preserving a reasonable degree of freedom in future choices.

Strotz [273] considers the maximization of utility in an additive, discounted form over a continuous-time future. Suppose a maximum-utility plan is selected at time t_1. Then, if utilities remain the same as time advances, except for the natural shift in the time-discount function, will the original plan continue to be the maximum-utility plan at time $t_2 > t_1$? In the case considered, Strotz shows that it will if and only if the future is discounted at a constant rate. If this is not so and the individual realizes it, his conflict may be resolved by one of two strategies, *precommitment* or *consistent planning*. Under precommitment he sticks to his original plan, even at a price, so as (for example) to enforce personal discipline and avoid diverting temptations. Under consistent planning he follows the plan that maximizes (future) utility at each instant. Then the only relevant part of his discount function is its instantaneous rate of change at the present moment. *Consistent planning* has the effect of replacing the individual's discount function by a constant-rate discount function. Strotz comments also on the future flexibility of choice problem discussed by Koopmans [161].

7. Utility Differences and Even-Chance Alternatives

Two closely-related notions in utility theory are comparable preference differences and even-chance alternative comparisons. Chipman [50, p. 216] credits Pareto [215, p. 264] and Frisch [115] for introducing the former notion into the literature; later contributions were made by Lange [170], Alt [8], Samuelson [240], and Weldon [308]. The even-chance alternative notion dates at least from Ramsey [231].

Both notions involve comparisons of ordered pairs (x, y) in $X \times X$. For comparable preference differences we use a relation $\lesssim\cdot$ on $X \times X$, with $(x, y) \lesssim\cdot (z, w)$ interpreted as: the directed difference in preference from x to y [associated with $u(x) - u(y)$] is not greater than that from z to w. For example, you may intuitively feel that your difference in preference between \$10 and \$0 is less than that between \$100 and \$10, whence $(\$10, \$0) \lesssim\cdot (\$100, \$10)$.[27]

Proposition 7.1. (Ordered Utility Differences). *A number $u(x)$ can be assigned to each x in X so that if x, y, z, w are in X then*

$$(7.1)\quad (x, y) \lesssim\cdot (z, w) \quad \textit{if and only if} \quad u(x) - u(y) \leq u(z) - u(w).$$

For even-chance alternatives we interpret (x, y) in $X \times X$ as an alternative that gives the individual an even chance of getting x or y (not both) when $x \neq y$. One may think of a chance event, such as getting "heads" on the flip of

[27] A different method of measuring the utility of money, based on such intensity-of-feeling questions as: "How much money would I have to give you to make you *twice as happy* as you would be if I were to give you \$10.00?" is discussed by Galanter [117] and Stevens [267, 268]. If the answer is \$45.00 (which comports with Galanter's findings), set $u(\$45) = 2u(\$10)$. See also Torgerson [290] and Stevens [269] for discussion of related matters.

a coin,[28] such that the individual gets x from (x, y) if the event occurs or y if the event does not occur. For even-chance alternatives we use $\lessdot$ on $X \times X$, interpreting $(x, y) \lessdot (z, w)$ as: (x, y) is not preferred to (z, w). You may feel, for example, that (\$0, \$100) $\lessdot$ (\$50, \$80).

Proposition 7.2 (Even-Chance Utilities). *A number $u(x)$ can be assigned to each x in X so that if x, y, z, w are in X then*

$$(7.2) \quad (x, y) \lessdot (z, w) \quad \textit{if and only if} \quad u(x) + u(y) \leqq u(z) + u(w).$$

If (7.1) holds and $(x, y) \leqq\cdot (y, x)$, then $u(x) \leqq u(y)$ and we therefore define $x \lessdot y$ to mean that $(x, y) \leqq\cdot (y, x)$. If (7.2) holds and $(x, x) \lessdot (y, y)$, then $u(x) \leqq u(y)$, so that we define $x \lessdot y$ to mean that $(x, x) \lessdot (y, y)$. It follows that both propositions imply Proposition 4.1.

If (7.2) holds, one can define $u(x, y) = \frac{1}{2}u(x) + \frac{1}{2}u(y)$, which equates the utility of an even-chance alternative to the average or expected utility of its two components. Generalization of the expected-utility notion is pursued in the next section.

The tie between (7.1) and (7.2) is simple: if $(x, y) \lessdot (z, w)$ as in (7.2), then $u(x) + u(y) \leqq u(z) + u(w)$. This can be rewritten as $u(x) - u(z) \leqq u(w) - u(y)$ which by (7.1) suggests $(x, z) \leqq\cdot (w, y)$. The procedure is reversible. Thus, *if* we define $(x, y) \lessdot (z, w)$ if and only if $(x, z) \leqq\cdot (w, y)$, then Propositions 7.1 and 7.2 are equivalent. *However*, as Weldon [308], Ellsberg [91], and others have pointed out, there is danger in this, for the two concepts are not equivalent. An individual might, for example, feel that (\$10,000, \$0) $\leqq\cdot$ (\$30,000, \$10,000) but prefer \$10,000 outright to an even-chance gamble between \$0 and \$30,000. Hence, *even though both propositions may hold for a person, the u functions in the two cases may be substantially different.*

Finite-Set Considerations

The groundwork for both (7.1) and (7.2) is given in Section 5 via the additivity theory for $n = 2$. Specifically, T_m is defined for ordered pairs (x, y) in $X \times X$ as it was defined for ordered pairs (x_1, x_2) in $X_1 \times X_2$. Assumptions 5.1_m and 5.1 are otherwise left intact in the even-chance context; in the comparable-difference context $\lessdot$ in 5.1_m is replaced by $\leqq\cdot$.

With the indicated alterations in interpretations we know from Section 5 that if X is a finite set then there are numerical functions u_1, u_2, v_1, v_2 on X such that

$$(7.1^*) \quad (x, y) \leqq\cdot (z, w) \quad \textit{if and only if} \quad u_1(x) + u_2(y) \leqq u_1(z) + u_2(w)$$

and

$$(7.2^*) \quad (x, y) \lessdot (z, w) \quad \textit{if and only if} \quad v_1(x) + v_2(y) \leqq v_1(z) + v_2(w)$$

if and only if $\leqq\cdot$ (or $\lessdot$) on $X \times X$ is connected and Assumption 5.1 holds.

[28] Davidson, Suppes, and Siegel [65] found that some people do not behave as if "heads" and "tails" are equally likely and suggest other chance events for the equally-likely purpose.

To get (7.1) from (7.1*) Scott [252] uses the following assumption, based directly on the notion of directed difference comparisons.

Assumption 7.1. (Directed Difference). *If x, y, z, w are in X and $(x, y) \leqq\cdot (z, w)$, then $(w, z) \leqq\cdot (y, x)$.*

On defining $u(x)$ for (7.1) by $u(x) = u_1(x) - u_2(x)$ for each x in X, Scott shows that (7.1) follows from (7.1*).

To get (7.2) from (7.2*) the following assumption, based directly on the equally-likely notion, is used.

Assumption 7.2 (Even-Chance). If x and y are in X, then $(x, y) \sim (y, x)$.[29]

On defining $u(x)$ for (7.2) by $u(x) = v_1(x) + v_2(x)$ for each x in X, it is easy to show that (7.2) follows from (7.2*).

According to language originating with Coombs [56], a utility function u on X satisfying (7.1) is called a higher ordered metric measure of utility.[30] In both (7.1) and (7.2) utilities are unique up to transformations that preserve the ordering of differences.

The development of necessary and sufficient conditions for Proposition 7.1 (connectivity, Assumptions 5.1 and 7.1) and Proposition 7.2 (connectivity, Assumptions 5.1 and 7.2) when X is finite has an interesting history, much of which is recounted by Luce and Suppes [183, Section 2.4]. A number of implications of Assumption 5.1_m for small m have been used, but it seems that the denumerable Assumption 5.1 is required in any non-existential set of assumptions sufficient for (7.1) or (7.2) when X is finite.

Infinite-Set Considerations

Suppes and Winet [279] present a list of assumptions sufficient for (7.1) that require X to be infinite. They work with absolute differences instead of directed differences. Each set of assumptions cited in the second paragraph under "Additive Utilities," Section 5, for $n = 2$, implies Proposition 7.1 [or 7.2] when $X_1 \times X_2$ is replaced by $X \times X$, $\preccurlyeq$ is replaced by $\leqq\cdot$ for (7.1), and Assumption 7.1 [or 7.2] is added. Debreu [71, 72] and Pfanzagl [216, 217] also present assumptions that imply Proposition 7.2. Each theory cited or alluded to in this paragraph requires X to be infinite, and u on X satisfying (7.1) or (7.2) is "unique up to linear, order-preserving transformations" as in (5.5).

8. Expected Utilities

This section applies $\preccurlyeq$ to $\mathcal{P}$, which is the set of simple probability distributions on a set of consequences X. Each simple probability distribution P in $\mathcal{P}$ is a function that assigns a nonnegative number $P(x)$ to each x in X so that (1) all but a finite number[31] of x in X have $P(x) = 0$, (2) the sum[32] of the $P(x)$

[29] A similar assumption appears on p. 28 of Davidson, Suppes, and Siegel [65].

[30] Ordered metric measures are discussed by Coombs [57, 59], Aumann and Kruskal [24], Becker and Siegel [34, 35], Coombs and Beardslee [60], Farrer [93], Fishburn [96], Hurst and Siegel [146], Luce and Suppes [183], Suzuki [281], Siegel [258], Shepard [257], and Torgerson [290].

[31] Cramer [62] gives assumptions for the expected-utility model for the set of simple distributions that have $P(x) > 0$ for at most two x's in X. Blackwell and Girshick [40]

that are positive equals 1. When $x \neq y$, the even-chance alternative (x, y) in the previous section corresponds to the P for which $P(x) = P(y) = \frac{1}{2}$.

For a specific P in $\mathcal{P}$ suppose P (get \$0) = .3, P (get \$10) = .6, and P (get \$100) = .1. Then, if an individual "selects" P, he will get exactly one of the three amounts, \$10 being twice as likely as \$0 and six times as likely as \$100.

If P and Q are in $\mathcal{P}$ and p is a number between 0 and 1 inclusive, then P^*, defined by $P^*(x) = pP(x) + (1 - p)Q(x)$ for each x in X, is also a probability distribution in $\mathcal{P}$. If $P(\$0) = .3$, $P(\$10) = .7$, $Q(\$0) = .5$, $Q(\$50) = .5$ and $p = \frac{1}{2}$, then $P^*(\$0) = .4$, $P^*(\$10) = .35$, $P^*(\$50) = .25$. For simplicity we shall write $P^* = pP + (1 - p)Q$, bearing in mind that *$pP + (1 - p)Q$ is a distribution in $\mathcal{P}$, not a number.*

Proposition 8.1 (Expected Utilities). *A number $u(P)$ can be assigned to each P in $\mathcal{P}$ so that if P and Q are in $\mathcal{P}$ and $0 \leqq p \leqq 1$, then*

$$P \preccurlyeq Q \quad \textit{if and only if} \quad u(P) \leqq u(Q) \tag{8.1}$$

$$u(pP + (1 - p)Q) = pu(P) + (1 - p)u(Q). \tag{8.2}$$

The function u on $\mathcal{P}$ is extended to include X by defining $u(x) = u(P)$ when $P(x) = 1$, and $x \preccurlyeq y$ means that $u(x) \leqq u(y)$. Using (8.2) it can be shown that if P is a distribution in $\mathcal{P}$ that has probabilities $P(x^1), P(x^2), \cdots, P(x^m)$ for $x^1, x^2, \cdots, x^m$ in X with $P(x^1) + P(x^2) + \cdots + P(x^m) = 1$, then (with the x^i all different)

$$u(P) = P(x^1)u(x^1) + P(x^2)u(x^2) + \cdots + P(x^m)u(x^m). \tag{8.3}$$

This says that if Proposition 8.1 holds, then the utility of any P in $\mathcal{P}$ can be computed as the weighted sum of the utilities of the x in X, the weights being the probabilities assigned by P. Equation (8.3) [or (8.2)] is an expected-utility equation. Coupled with (8.1) it asserts that a distribution with a higher expected utility is preferred to a distribution with a smaller expected utility. If $P(\$0) = .5$, $P(\$1000) = .5$, $Q(\$500) = 1$, and $u(\$0) = 0$, $u(\$1000) = 10$, $u(\$500) = 4$, then P is preferred[33] to Q since $u(P) = .5(0) + .5(10) = 5$ and $u(Q) = 4$.

A functon u satisfying (8.1) and (8.2) is "unique up to linear, order-preserv-

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consider the case where all but a finite or denumerable number of $P(x)$ equal 0, which "generalizes" the theory given below by extending Assumption 8.2. See also Chernoff and Moses [49] for an elementary treatment. Arrow [17] and Fishburn [107] extend the theory further with assumptions for expected utility for more general probability measures. Luce [181] presents assumptions that generate the probabilities as well as the utilities for the model described by (8.1) and (8.3).

[32] The probability assigned by P to any subset of X equals the sum of the probabilities of the elements in the subset. To be technically proper, a probability distribution or measure is a function defined on a specified set of subsets (of X), with certain properties. Our simplified characterization will suffice for the present discussion.

[33] Mosteller and Nogee [206] were first in trying to measure u experimentally in this context. The prior Preston-Baratta experiment [223] offers evidence that subjects distort given "objective" probabilities, and later experiments by Edwards [81, 83, 84, 85] and others support this conclusion. Edwards [82, 86], Luce and Suppes [183], and Becker and McClintock [33] give extensive reviews of experimental findings. See also Coombs, Bezembinder, and Goode [61] for an interesting and recent test of expectation theories.

ing transformations": if a specific function u on $\mathcal{P}$ satisfies (8.1) and (8.2), then so does another function v on $\mathcal{P}$ if and only if there are numbers a and b with $a > 0$ such that $v(P) = au(P) + b$ for all P in $\mathcal{P}$.

If $\preccurlyeq$ on $\mathcal{P}$ (in place of X) satisfies Assumptions 4.1 and 4.2, then there is utility function u on $\mathcal{P}$ satisfying (8.1). To show that there is at least one such function that also satisfies the expectation property (8.2), something additional is required (namely Assumption 8.2 below).

A number of authors[34] have stated assumptions that imply Proposition 8.1. Because Daniel Bernoulli [37] and later von Neumann and Morgenstern [305] pioneered in this, expected utilities are often called Bernoullian utilities or von Neumann-Morgenstern utilities.[35] The following three assumptions, *necessary and sufficient* for Proposition 8.1, are fairly standard.

Assumption 8.1.[36] (Weak Order). *$\preccurlyeq$ is a weak order on $\mathcal{P}$.*

Assumption 8.2 (Sure-Thing). *If P, Q, and R are in $\mathcal{P}$, and if $P \prec Q$ and $0 < p < 1$, then $pP + (1 - p)R \prec pQ + (1 - p)R$.*

Assumption 8.3 (Archimedean). *If P, Q and R are in $\mathcal{P}$, and if $P \prec Q$ and $Q \prec R$, then there are numbers p, q both strictly between 0 and 1 such that $pP + (1 - p)R \prec Q$ and $Q \prec qP + (1 - q)R$.*

In the last assumption we could expect p to be fairly large (near 1) and q to be fairly small (near 0). Together the assumptions imply that if $P \prec Q \prec R$, then there is a unique p between 0 and 1 such that $Q \sim pP + (1 - p)R$. Given $u(P)$ and $u(R)$, $u(Q)$ is determined by the equation $u(Q) = pu(P) + (1 - p)u(R)$.

As common-sense guidelines for computing preferences, the assumptions have not gone unchallenged. Aumann [21] challenges the connectivity part of weak order and works with a quasi order (Section 4). Violation of Assumption 8.3 (see Thrall [286]) leads to lexicographic expected utilities, discussed in the next section.

An ingenious example due to Allais [6] (see Savage [246, pp. 101–103]) challenges Assumption 8.2, which we have termed "sure-thing" after Savage. Samuelson [243] calls this an independence assumption, since it is vital to the multiplicative-additive form for u given by (8.2).[37] It differs of course from the independence assumptions in Section 5, but Tversky [299] formulates a general independence (or cancellation) condition that implies Assumption 5.1 and Assumption 8.2 in their respective contexts.[38]

[34] Including von Neumann and Morgenstern [305], Friedman and Savage [113, 114], Marschak [190], Herstein and Milnor [140], Luce and Raiffa [182], and Jensen [151].

[35] Adams [2] presents an exposition of expected utility theory and Swalm [282] gives a lucid, nonmathematical discussion.

[36] Jensen [151] shows that Assumptions 8.1, 8.2, and 8.3 imply that if $P \sim Q$ and $0 \leq p \leq 1$, then $pP + (1 - p)R \sim pQ + (1 - p)R$. See also Samuelson [243] and Malinvaud [186].

[37] Chipman [50, p. 219] makes a penetrating comment on this point.

[38] See also Tversky [296, 297, 298] for recent behavioral tests of the independence assumption in the expected-utility context.

Expected Utility of Money

A merchant considers insuring a cargo. He figures the ship is nine times more likely to deliver than sink. Without insurance he makes \$10,000 on delivery or loses \$50,000 if the ship sinks; his expected profit is .9(\$10,000) + .1(−\$50,000) = \$4,000. Insurance costs \$4,000 and pays \$30,000 on sinkage. With insurance his expected profit is .9(\$6,000) + .1(−\$24,000) = \$3,000. Would he be foolish to purchase the insurance?

In 1738 Daniel Bernoulli [37] argued that, even though the merchant's expected profit is less with insurance, he cannot be called foolish for buying it. Such examples led Bernoulli to propose the policy of maximizing expected utility rather than expected profit. If the merchant's utilities of additions of \$10,000, −\$50,000, \$6,000, and −\$24,000 to his present wealth were 2, −20, 1, and −6, his expected utility without insurance would be $.9(2) + .1(-20) = -.2$ and with insurance would be $.9(1) + .1(-6) = .3$.

Bernoulli speculated that an individual's utility of wealth increases at a decreasing rate of increase as wealth increases.[39] Recent authors, including Friedman and Savage [113, 114], Markowitz [188], and Friedman [112] argue that the diminishing rate of increase does not fully agree with observed behavior and propose different functions for the utility of wealth in the expected-utility context.[40]

In prescriptive application we should bear in mind that a decision maker's utility of money (or anything else) is conditioned by his present situation, requirement for decision, implications his decision has for the future, *and* his attitude toward risk-taking. The theory given here does not attribute special properties to an individual's utility function for money such as continuity[41] and differentiability. The simplifying assumption "if x and y are amounts of money, $0 \leq p \leq 1$, and if $Q(px + (1 - p)y) = 1$ and $P(x) = p$, $P(y) = 1 - p$, then $Q \sim P$" does imply, in the context of Proposition 8.1, that u on X is continuous, differentiable, and linear: $u(x) = ax + b$ for all x in X. In this case, maximization of expected utility and maximization of expected profit amount to the same thing. Schlaifer [250, 251] argues that this is not an unreasonable assumption in decisions that involve only a fraction of a business' assets.

Pfanzagl [217] considers a different assumption he calls the consistency axiom.[42] In a slightly different form than Pfanzagl uses, it says that "if $P(\$x + \$y) = Q(\$x)$ for all x in X and if $Q \sim \$z$, then $P \sim \$z + \$y$." If u on X increases in x and is continuous, this implies that u on X (in the context of Proposition 8.1) is either linear or has the form $u(x) = ak^x + b$ with $a > 0$ if $k > 1$ and $a < 0$ if $0 < k < 1$. A similar result is derived by Pratt [220].

[39] For example, $u(\$0) = 0$, $u(\$1000) = 10$, $u(\$2000) = 19$, $u(\$3000) = 27$, and so forth. Economists refer to this as the diminishing marginal utility of money.

[40] Edwards [82, pp. 393–394] and Robertson [239, pp. 672–675] give brief summaries. Attitudes toward risk are discussed also by Pratt [220] and Swalm [282]. See also Stevens [267, 268] and Galanter [117].

[41] If he gets a bonus for making \$1,000,000 net profit, his utility for net profit may jump at the \$1,000,000 point, reflecting the incentive purpose of the bonus.

[42] Krantz and Tversky [167] comment on this.

Expected utility of course applies to things other than money. The next section considers cases where a number of factors influence preferences.

9. Expectations and Multidimensional Consequences

Let $\mathcal{P}$ be the set of all simple probability distributions on a set of consequences X (Section 8), and let X be equal to or a subset of the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ (Section 5). In addition, let $\mathcal{P}_i$ be the set of all simple probability distributions on the i^{th} factor X_i, $i = 1, 2, \cdots, n$. Given P in $\mathcal{P}$ we identify P_i in $\mathcal{P}_i$ as the marginal distribution of P on X_i. If $n = 2$, $P(x_1, x_2) = .1$, $P(x_1, y_2) = .3$, and $P(y_1, y_2) = .6$, then $P_1(x_1) = .1 + .3 = .4$, $P_1(y_1) = .6$, and $P_2(x_2) = .1$, $P_2(y_2) = .3 + .6 = .9$.

Proposition 9.1 (Additive, Expected Utilities). *Proposition* 8.1 *holds, and a number $u_i(P_i)$ can be assigned to each P_i in $\mathcal{P}_i$, $i = 1, 2, \cdots, n$ so that if P is in $\mathcal{P}$ and if $P_1, P_2, \cdots, P_n$ are the marginal distributions of P on $X_1, X_2, \cdots, X_n$ respectively, then*

$$u(P) = u_1(P_1) + u_2(P_2) + \cdots + u_n(P_n) \tag{9.1}$$

with u on $\mathcal{P}$ satisfying (8.1) *and* (8.2).

Defining $u_i(x_i) = u_i(P_i)$ when $P_i(x_i) = 1$, Proposition 9.1, if valid, says we can compute $u(P)$ by summing the expected utilities $u_i(P_i)$ of the marginal distributions: if $P_i(x_i^1) + P_i(x_i^2) + \cdots + P_i(x_i^m) = 1$, then $u_i(P_i) = P_i(x_i^1)u_i(x_i^1) + P_i(x_i^2)u_i(x_i^2) + \cdots + P_i(x_i^m)u_i(x_i^m)$.

Proposition 9.2 (Lexicographic, Expected Utilities). *A number $u_i(P_i)$ can be assigned to each P_i in $\mathcal{P}_i$ for $i = 1, 2, \cdots, n$ so that if P and Q are in $\mathcal{P}$, if $P_1, P_2, \cdots, P_n$ $[Q_1, Q_2, \cdots, Q_n]$ are the marginal distributions of $P[Q]$ on $X_1, X_2, \cdots, X_n$ respectively, and if $0 \leqq p \leqq 1$, then*

$$P \lessdot Q \quad \textit{if and only if} \quad (u_1(P_1), u_2(P_2), \cdots, u_n(P_n)) \overset{L}{\leqq} (u_1(Q_1), u_2(Q_2), \cdots, u_n(Q_n)) \tag{9.2}$$

$$u_i(pP_i + (1 - p)Q_i) = pu_i(P_i) + (1 - p)u_i(Q_i) \quad \textit{for each} \quad i. \tag{9.3}$$

The definition of the lexicographic order relation $\overset{L}{\leqq}$ is given in Section 5. In (9.3) $pP_i + (1 - p)Q_i$ is the distribution in $\mathcal{P}_i$ that assigns probability $pP_i(x_i) + (1 - p)Q_i(x_i)$ to each x_i in X_i. The expected utility $u_i(P_i)$ is computed the same way in both Propositions 9.1 and 9.2, as described above.

Independence Assumption

The independence assumption for the multidimensional aspect of the consequences is closely related to Assumption 5.1. In the present context the general independence assumption is:

Assumption 9.1 (Independence). *If P and Q are in $\mathcal{P}$, if P_i $[Q_i]$ is the marginal distribution of $P[Q]$ on X_i, and if $P_i = Q_i$ for $i = 1, 2, \cdots, n$, then $P \sim Q$.*

This is implied both by Proposition 9.1 and by 9.2. If $P_i = Q_i$ for each i, then $u_i(P_i) = u_i(Q_i)$ for each i. If (9.1) holds, then $u(P) = u(Q)$ so that $P \sim Q$ by (8.1). If (9.2) holds, then $P \lessdot Q$ and $Q \lessdot P$ so that $P \sim Q$. The

effect of the present independence assumption is to permit preferences between probability distributions to be reckoned on the basis of their marginal distributions. It is definitely a simplifying assumption since there is no "common-sense argument" justifying its adoption in all situations. As in Section 6, item 7, let x_1, x_2 be respectively the meats you have for dinner tonight and tomorrow night and let P(steak, steak) $=$ P(lobster, lobster) $= .5$ and Q(steak, lobster) $=$ Q(lobster, steak) $= .5$. Since P_i(steak) $=$ P_i(lobster) $=$ Q_i(steak) $=$ Q_i(lobster) $= .5$ for $i = 1, 2$, $P_i = Q_i$ for $i = 1, 2$. But if you enjoy variety, it may well be true that $P < Q$, which violates Assumption 9.1.

Additive Utilities

Fishburn [97, 101, 104] has shown that *Proposition* 9.1 *is true if and only if Assumptions* 8.1, 8.2, 8.3 *and* 9.1 *are true, except* for the case in which $n \geq 3$ and X is an arbitrary subset of $X_1 \times X_2 \times \cdots \times X_n$ with an infinite number of elements.[43] When $X = X_1 \times X_2 \times \cdots \times X_n$, it has been shown [97] that Assumption 9.1 may be replaced by one of its consequences, namely: if x, y, z, w are in X and $P(x) = P(y) = \frac{1}{2}$ and $Q(z) = Q(w) = \frac{1}{2}$ and $P_i = Q_i$ for each i, then $P \sim Q$.

If Proposition 9.1 holds when $X = X_1 \times X_2 \times \cdots \times X_n$, then the u_i and u have the uniqueness properties of (5.3) and (5.5). However, if X is a subset of $X_1 \times X_2 \times \cdots \times X_n$, then (5.3) does not necessarily exhaust all permissible transformations of the u_i.

Additive, expected utilities have been used by Churchman, Ackoff, and Arnoff [55, pp. 150–152] and Raiffa and Schlaifer [228] among others. Phelps [218] and Hakansson [1?6] use additive, discounted expected utilities in a denumerable-period formulation.

Lexicographic Utilities

Necessary and sufficient conditions for Proposition 9.2 when $X = X_1 \times X_2 \times \cdots \times X_n$ are stated by Fishburn [106], and include Assumptions 8.1 and 9.1, a sure-thing assumption like 8.2, a lexicographic order assumption, and an Archimedean assumption for each dimension. In this case each u_i in (9.2) is "unique up to linear order-preserving transformations" of the form $v_i(P_i) = a_i u_i(P_i) + b_i$ for all P_i in $\mathcal{P}_i$, with $a_i > 0$. (Unlike Propositions 5.1 and 5.2, Propositions 9.1 and 9.2 are always mutually incompatible except in the trivial case where only one dimension or X_i has any effect on preferences.)

Hausner [139] gives a more general development of lexicographic, expected utilities[44] which does not assume a product structure for X. He shows that Assumptions 8.1, 8.2, and the indifference ($\sim$) version of 8.2 imply lexicographic, expected utilities whose dimensionality equals one if and only if Assumption 8.3

[43] We conjecture that the statement is true also for this case, but an explicit proof is missing. Tversky's general theory [299] appears to cover this exception.

[44] The utility vector for P would be written $(u_1(P), u_2(P), \cdots, u_n(P))$ instead of $(u_1(P_1), u_2(P_2), \cdots, u_n(P_n))$.

also holds, in which case his result reduces to Proposition 8.1. Additional contributions on this subject are made by Chipman [50, Section 3.5].

10. Expected Utility and Subjective Probability

The theory of Section 8 presupposes knowledge of the probabilities of consequences associated with various courses of action. Because probabilities, like utilities, must be measured, there is interest in formulating assumptions that imply both numerical utilities and probabilities satisfying an expected-utility model.[45]

This section considers so-called subjective probability[46] in the expected-utility context. Such a probability is a measure of the confidence an individual places in the truth of a proposition such as "it will rain tomorrow" or "if our cost bid for the contract is $100,000, then we will get the contract". We will consider subjective probabilities based on preferences. A different approach, used, for example, by Koopman [158, 159], Kraft, Pratt, and Seidenberg [163], Scott [252, Section IV], Villegas [302, 303], Luce [180], and to some extent by Good [122] and de Finetti [74], makes assumptions based directly on a relation "is not more probable than".[47] Luce and Suppes [183, pp. 295–298] review some of this theory.

Because it is widely discussed and meshes with previous material in this paper, the following model will be used.[48] A decision maker is to select an act (course of action) from a set F of acts with elements $f, g, \cdots$. The act selected will result in the occurrence of exactly one consequence (unknown before hand) in X.[49] The resulting consequence depends on which state in a set S of states with elements $s_1, s_2, \cdots, s_n$ is the "true state." The decision maker does not know which state is the true state. If the resulting consequence depends on the action of some other person(s) [whether or not the road has been booby-trapped] determined independently of the act selected by our decision maker, these actions could constitute the states in S. If the consequence depends on a system's actual physical state [whether or not these mushrooms are poisonous] that is not affected by the

[45] Other approaches to decision making under uncertainty have been proposed. See, for example, Wald [306, 307], Savage [245, 246], and, for summaries, Luce and Raiffa [182, Chapter 13], Miller and Starr [201, Chapter 5], Hall [137, Chapter 11], Ackoff [1, pp. 50–61], Milnor [202], and Fishburn [103].

[46] Other interpretations of probability are discussed by Nagel [211], Keynes [155], von Mises [304], Reichenbach [235], Carnap [46, 47], Good [122, 123], Savage [246], Ramsey [231], Fishburn [96, Chapter 5], Hacking [134], de Finetti [75], and Georgescu-Roegen [119].

[47] In terms of *preferences* "event A is not more probable than event B" means that "for all x, y such that $x \preccurlyeq y$, the alternative resulting in y if A occurs and x if A doesn't occur is not preferred to the alternative resulting in y if B occurs and x if B doesn't occur".

[48] Under a natural correspondence of probabilities this model is equivalent to the following: for any two acts f and g, $f \preccurlyeq g$ if and only if $u(f) \leq u(g)$ with $u(f) =$ [Sum of $P(x \mid f)u(x)$ over all x in X], where $P(x \mid f)$ is the probability that x will result if f is implemented. See Luce [181] for assumptions for the latter model, Jeffrey [149, 150] for discussion of another related model, and Sneed [261] for a critique of Jeffrey's model.

[49] Elements in X can depend on aspects of the acts as well as outcomes that might follow therefrom.

act selected by the decision maker, the physical states could constitute the states in S. $p(s_i)$ is the decision maker's probability for the proposition "s_i is the true state", and $f(s_i)$ is the consequence in X that will result if s_i is the true state and act f is implemented.

Letting the states $s_1, s_2, \cdots, s_n$ be fixed, we represent each act f in F as an n-tuple $(x^1, x^2, \cdots, x^n)$ of consequences in X, with $f(s_i) = x^i$ for each i. That is, $(x^1, x^2, \cdots, x^n)$ *is an act in* F *that results in* x^i *if* s_i *is the true state*, $i = 1, 2, \cdots, n$.[50] The set of acts F is then equal to or a subset of $X \times X \times \cdots \times X$ (n times).

Proposition 10.1 (Expected Utility-Subjective Probability). *A number* $u(x)$ *can be assigned to each* x *in* X *and a nonnegative number* $p(s_i)$ *can be assigned to each state* s_i *in* S *so that if* $(x^1, x^2, \cdots, x^n)$ *and* $(y^1, y^2, \cdots, y^n)$ *are acts in* F, *then*

$$(10.1)\quad (x^1, x^2, \cdots, x^n) \prec (y^1, y^2, \cdots, y^n) \quad \textit{if and only if}$$
$$p(s_1)u(x^1) + p(s_2)u(x^2) + \cdots + p(s_n)u(x^n)$$
$$\leq p(s_1)u(y^1) + p(s_2)u(y^2) + \cdots + p(s_n)u(y^n).$$

To prohibit the uninteresting possibility that $p(s_i) = 0$ for all i, we shall henceforth assume that $f < g$ for at least one pair of acts in F. From (10.1) the utility of act $(x^1, x^2, \cdots, x^n)$ may be defined as $u(x^1, x^2, \cdots, x^n) = p(s_1)u(x^1) + p(s_2)u(x^2) + \cdots + p(s_n)u(x^n)$. If the $p(s_i)$ are normalized (if necessary) so that $p(s_1) + p(s_2) + \cdots + p(s_n) = 1$ and if $(x, x, \cdots, x)$ is a constant act[51] in F, then $u(x, x, \cdots, x) = u(x)$.

Finite-State Considerations

As in (10.1) let S have n states. If X and $X_1 \times X_2 \times \cdots \times X_n$ in Section 5 are replaced by F and $X \times X \times \cdots \times X$ (n times), respectively, then each additivity theory cited in Section 5 implies that there are functions $u_1, u_2, \cdots, u_n$ each defined on X such that if $(x^1, x^2, \cdots, x^n)$ and $(y^1, y^2, \cdots, y^n)$ are in F then

$$(10.2)\quad (x^1, x^2, \cdots, x^n) \prec (y^1, y^2, \cdots, y^n) \quad \text{if and only if}$$
$$u_1(x^1) + u_2(x^2) + \cdots + u_n(x^n) \leq u_1(y^1) + u_2(y^2) + \cdots + u_n(y^n).$$

The independence assumptions of Section 5, applied to $\prec$ on F with $X \times X \times \cdots \times X$ (n times) replacing $X_1 \times X_2 \times \cdots \times X_n$, convey the notion that the s_i are independent: that no more that one s_i is the true state. Violation of independence in this sense may require a reformulation of the states.

Expression (10.2) is one way of approaching (10.1). If additional assumptions can be stated that permit us to write $u_i(x) = p(s_i)u(x)$, $p(s_i) \geq 0$ in all cases,

[50] The notion that one of the s_i is believed certain to be the true state is conveyed by the assumption: $(x^1, x^2, \cdots, x^n, x) \sim (x^1, x^2, \cdots, x^n, y)$ for all acts $(x^1, x^2, \cdots, x^n)$ in F and consequences x, y in X, where the $n + 1^{st}$ component (x or y) is the consequence resulting if no s_i is the true state.

[51] An act resulting in the same consequence regardless of which state is the true state.

then (10.1) follows from (10.2). To the best of my knowledge, necessary and sufficient conditions for (10.1) that are stated solely in terms of $\preccurlyeq$ on F have not been given for X either finite or infinite when F is an arbitrary subset of $X \times X \times \cdots \times X$ (n times). Partly because of this and to simplify matters, I shall tentatively assume in what follows that $F = X \times X \times \cdots \times X$ (n times).

The notion that the s_i are considered equally likely is imparted by a permutation assumption: if $(x^1, x^2, \cdots, x^n)$ is a permutation of $(y^1, y^2, \cdots, y^n)$, then $(x^1, x^2, \cdots, x^n) \sim (y^1, y^2, \cdots, y^n)$. Define $u(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$. Then (10.2) and the permutation assumption imply that $(x^1, x^2, \cdots, x^n) \preccurlyeq (y^1, y^2, \cdots, y^n)$ if and only if $u(x^1) + u(x^2) + \cdots + u(x^n) \leq u(y^1) + u(y^2) + \cdots + u(y^n)$.[53] In this case we may let $p(s_i) = 1/n$ for each i in (10.1).

Things get a bit sticky when the s_i are not considered equally likely. Thus far only[54] sufficient conditions (not all necessary) have been stated for (10.1), mainly with the aid of artifices that are not naturally part of the situation. Anscombe and Aumann [11] obtain (10.1) with a double application of the expected utility theory of Section 8, applying it once to the set of finite probability distributions $\mathcal{R}$ defined on X and once to the set of finite probability distributions $\mathcal{R}^*$ defined on $\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}$ (n times). The probabilities used in the Section 8 theory are associated with the outcomes of chance experiments on symmetric gambling devices such as well-balanced roulette wheels. The $p(s_i)$ are derived from the utility functions obtained from the double application.

In a closely-related development Fishburn [108] applies the theory of Section 8 to the set of horse lotteries [$\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}$], then uses an additional order assumption for invariance of preferences under each state to derive (10.1). This paper also shows that (10.1) can be obtained on the basis of simple even-chance alternatives of the form (f, g) [f and g are equally likely to result] when Debreu's theory [71, 72] is used as a starting point. Davidson and Suppes [64] derive a slightly different model than (10.1) by using the even-chance idea with a finite set of consequences that are assumed to be equally spaced in utility.

The device of using even-chance alternatives is related to Ramsey's suggestions [231] for measuring utilities and subjective probabilities. This is discussed by Luce and Suppes [183, Section 3.3].

Pratt, Raiffa, and Schlaifer [221] use a canonical experiment of essentially equally-likely outcomes to measure utilities and subjective probabilities in the model of Proposition 10.1.

Infinite-State Considerations

In 1954 Savage [246] published assumptions applying $\preccurlyeq$ to a set F of acts that imply an expected utility model like (10.1). By using a sufficiently dense, infinite set of states S, he does not require the artifices discussed above, although

[53] Chernoff [48] and Luce and Raiffa [182, Section 13.4] discuss an alternative approach to this result.

[54] Tversky's general theory [299] apparently gives necessary and sufficient conditions, but it appears difficult to state these in terms of $\preccurlyeq$ on F in a simple way.

his axioms do imply that S can be partitioned into arbitrarily many equally-likely events. Although several points in his theory have been criticized,[54] it has stood up well as a prescriptive guide for decision making under uncertainty.

Savage's act f in F is a function assigning a consequence in X to each state in S. In dealing with S he uses general subsets of S, called events. Many of his assumptions are similar to ones discussed above. Several summaries of Savage's theory are available, including those by Luce and Raiffa [182, pp. 302–304], Fishburn [96, pp. 175–178],[55] and Luce and Suppes [183, pp. 298–299]. Fishburn [108] notes that Savage's axioms imply that utility is bounded; see also Arrow [19].

Suppes' theory [276], based on even-chance alternatives (f, g), is presented as an alternative to Savage's. Suppes does not require all conceivable acts, and his theory is applicable to either a finite or infinite S.

Experimentation

A decision maker, faced with selecting an act in F under uncertainty, has the opportunity to perform an experiment from a set of experiments that includes the "null experiment" (i.e., no experiment performed). Any experiment may give him additional information about which state in S is the true state. Which experiment should he perform?

The theory of expected utility and subjective probability, as applied to this situation by Schlaifer [250, 251], Raiffa and Schlaifer [228], Pratt, Raiffa, and Schlaifer [222], Savage [246, 249], and Hadley [135], among others, proposes the following answer. A *derived act* (strategy, decision rule) for a given experiment assigns a *terminal act* in F to each possible outcome of the experiment. A maximum expected-utility derived act is then determined, and the experiment associated with this derived act is performed.

The maximum expected-utility derived act for experiment E, considered from the present perspective, is computed as follows. Let $0_1, 0_2, \cdots, 0_m$ be the possible outcomes of E, and let $u(E, 0_j; f_j(s_i))$ be the (present) utility associated with performing E, observing 0_j, implementing f_j in F and having $f_j(s_i)$ result when s_i is the true state. Then, let $p(s_i \mid E, 0_j)$ be the decision maker's probability for the proposition "s_i is the true state if outcome 0_j results from E". If E is the null experiment,[56] $p(s_i \mid E, 0_j) = p(s_i)$. The expected utility associated with E if 0_j occurs and f_j is implemented equals the sum over i of $p(s_i \mid E, 0_j)u(E, 0_j; f_j(s_i))$. We find the f_j that maximizes this sum; call it f_j^*. Then the derived act

[54] See, for example, Suppes [277, 278] and Ellsberg [92]. Raiffa [227] rebuts the latter. Savage notes disturbances of his own as he unfolds the theory in his book.

[55] Footnote 70, p. 177, is incorrect.

[56] In many cases $p(s_i \mid E, 0_j)$ is computed using Bayes' Theorem,

$$p(s_i \mid E, 0_j) = p(s_i)p(0_j \mid E, s_i)/p(0_j \mid E),$$

where $p(0_j \mid E, s_i)$ is the probability that 0_j will result when E is performed and s_i is the true state and $p(0_j \mid E) =$ [Sum over i of $p(s_i)p(0_j \mid E, s_i)$]. $p(0_j \mid E)$ is the probability that 0_j will occur if E is used.

f^* that assigns f_j^* to 0_j for $j = 1, 2, \cdots, m$ is a maximum expected-utility derived act for E. The expected utility[66] of f^* equals the sum over i and j of $p(s_i)p(0_j \mid E, s_i)u(E, 0_j; f_j^*(s_i))$.

Because of the amount of measurement and computation implied by this brief description, numerous simplifying assumptions are made by most people interested in applying the procedure. For example, almost everyone assumes that utilities of experimentation and terminal action are additive: $u(E, 0_j; f_j(s_i)) = u_1(E, 0_j) + u_2(f_j(s_i))$, and special functional forms are often used for u_1 and u_2. For further discussion see the references cited above and explanatory papers by, for example, Savage [247, 248], Edwards, Lindman, and Savage [88], Anscombe [9, 10], and Green [130].

11. Social Choice and Individuals' Preferences

The question of how a society or group shall choose among social alternatives has been of interest for several millennia, and the number of methods devised to answer it has been rather large. In concluding our discussion it seems appropriate to mention several proposals that relate individuals' preferences to social choice or group decision.

Arrow [18] and Rothenberg [240] provide extensive discussions. Arrow recommends Barbut [27] for an elementary exposition, Guilbaud [133] for a somewhat longer exposition, and Riker [237] for a summary (as of 1961). Luce and Raiffa [182, Chapter 14] is useful also. Black's book [39] contains his important contributions to the theory of elections and a delightful history of the subject that includes proposals by Borda (1733–1799), Laplace (1749–1827), Galton (1822–1911), Nanson (1850–1936), and especially Condorcet (1743–1794) and C. L. Dodgson (Lewis Carroll: 1832–1898). A psychology-oriented discussion by Coombs [59, Chapter 18] parallels and goes beyond some of Black's work.

For a very brief indication of some of the work in this area, let $\precsim_i$ ($i = 1, 2, \cdots, n$) denote the i^{th} individual's preference order (connected and transitive) on a set X of at least three social alternatives (states, candidates). Let $\precsim_s$ denote a "social order" (connected and transitive) on X. Following Arrow [18, p. 23] we define a *social welfare function* as *a process that assigns a social order* $\precsim_s$ *to each possible set of individual orders* $\precsim_1, \precsim_2, \cdots, \precsim_n$.

Two early proposals concerning social welfare functions are majority rule and the method of marks.

1. *Majority Rule* (Condorcet). Compare each pair of alternatives and let $x \leqq_M y$ mean that at least as many individuals prefer y to x as prefer x to y. [As usual, $x <_M y = (x \leqq_M y$ and not $y \leqq_M x)$.] If $\leqq_M$ is transitive, let $\precsim_s = \leqq_M$. If $n = 3$ and $x <_1 y <_1 z$, $y <_2 z <_2 x$, and $z <_3 x <_3 y$, then $x <_M y$, $y <_M z$, and $z <_M x$ showing that $\leqq_M$ need not be transitive. Condorcet was quite aware of this and Dodgson used the term "cyclical majorities" to describe it.[67] The example given is called the "voters' paradox" although the use of

[67] For recent comments on the frequency and significance of cyclical majorities see Campbell and Tullock [45], Klahr [156], Tullock [294] and DeMeyer and Plott [76].

"paradox" is questionable. The outcome of an election by successive elimination (compare two candidates at a time and eliminate the loser from further consideration) may depend on the order in which candidates are presented when cyclical majorities are present: the candidate presented last has the greatest advantage under successive elimination.

Let each voter's most preferred candidate be his "ideal" (Coombs). Black shows that if the number of voters is odd ($n = 3, 5, 7, \cdots$) and if the candidates or alternatives can be represented as points on a line such that, going left to right on the line, the preference of each voter increases up to his ideal point and decreases thereafter,[58] then $\leq_M$ is transitive. The candidate with a simple majority over each other candidate is the ideal of the median voter (whose ideal lies in the middle of the ideal points). Under slightly stronger conditions Coombs [58] and Goodman [125][59] show that $\leq_M$ coincides with the $\preccurlyeq_i$ of the median voter.

If the domain of Arrow's social welfare function (that is, the admissible sets of individual orders) is suitably restricted (for example, to those that satisfy Black's conditions under some arrangement of the alternatives along the line), then $\leq_M$ may be considered an appropriate definition of $\preccurlyeq_s$.

2. *Method of Marks* (Borda, Laplace). With m alternatives, assuming no ties (indifference) for any voter, assign the numbers (marks) $1, 2, \cdots, m$ to the least preferred, next least preferred, $\cdots$, most preferred alternative respectively for each voter. Add the marks for each alternative, and define $\preccurlyeq_s$ to coincide with the order of the totals. This is a special case of the general additive procedure in which a utility function u_i satisfying (4.1) is used for the i^{th} individual and the "social utility" of alternative x, say $u_s(x)$ is defined by $u_s(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$.

An additive-utility method does not necessarily yield the same $\preccurlyeq_s$ as does $\leq_M$ when $\leq_M$ is transitive. Condorcet published an example[60] in 1785 in which $\leq_M$ is transitive and the candidate having a simple majority over each other candidate cannot be selected by any additive method assigning marks of $a_1, a_2, \cdots, a_m$ to the m candidates for each voter by order of preference, with $a_1 < a_2 < \cdots < a_m$.

Two Objections Raised on Additive-Utility Methods

1. If a candidate is removed from the voting and the marks (1, 2, etc.) are revised as though he had never been present, then the social ordering of the other candidates may change: the original "winner" may lose out after the revision. This is described by saying that the procedure is not "independent of irrelevant alternatives". Proposals to get around it are often based on "strengthening" each u_i so that the removal of any x in X will leave the other utilities for

[58] As might be true in considering minimum-wage legislation. We imagine that each legislator has an ideal minimum wage and that his preference drops in either direction away from his ideal.

[59] See Luce and Raiffa [182, Section 14.7] for further discussion.

[60] Black [39, pp. 176–177].

the i^{th} individual unchanged. Goodman and Markowitz [126] and Minas and Ackoff [203], among others, consider this approach.[61] Doubts on its validity are raised by Arrow [18, pp. 9–11] and others.

2. The second objection concerns interpersonal comparisons of utility. Can any significant meaning be attached to the sum of different individuals' utilities? If so, what particular u_i functions should be used in defining

$$u_s(x) = u_1(x) + \cdots + u_n(x)?$$

Like the problem of cyclical majorities, the question of the normalization or alignment of individual utility functions has not been satisfactorily resolved. Some people feel that it is a meaningless question.

Arrow's Conditions

Arrow [18] considers the possibility of a social welfare function as defined above that satisfies the following apparently reasonable conditions

1. All logically possible orderings of the $\preccurlyeq_i$ are admissible.[62]

2. (Independence of irrelevant alternatives). Social preference or indifference between x and y in X shall depend solely on the individuals' preferences between x and y, without regard to preferences involving other alternatives.

3. (Pareto principle). If $x <_i y$ for $i = 1, 2, \cdots, n$, then $x <_s y$.

4. (Nondictatorship). There is no individual i such that, for all x, y in X, $x <_i y$ implies $x <_s y$ regardless of the preferences of the other individuals.

Arrow (pp. 98–100) proves that, when there are at least two individuals and three social alternatives, *there is no social welfare function satisfying these four conditions*. For example, any social welfare function satisfying conditions 1, 2, and 3 *must* be dictatorial.

If condition 1 is modified to satisfy Black's conditions as discussed above, then the method of majority rule satisfies the revised condition 1 plus conditions 2, 3 and 4 [Arrow: Theorem 4, p. 78].

A number of proposals[63] that modify Arrow's conditions and/or propose other conditions have been offered since the appearance of Arrow's first edition (1951). Some of these are discussed by Luce and Raiffa [182, Chapter 14] and Arrow (pp. 100–103). Criticisms of Arrow's formulation of the problem of social choice have been voiced by Little [174], Bergson [36], Kemp [154], Buchanan [43], Mishan [204], and Tullock in Appendix 2 of Buchanan and Tullock [44]. In reply, Arrow (pp. 103–108) argues "... that these criticisms are based on misunderstandings of my position and indeed of the full implications of the critics' own positive views ... all implicitly accept the essential formulation stated here: The social

[61] See Luce and Raiffa [182, Section 14.6], for further discussion.

[62] An observation by Blau [41] on condition 1 as originally stated by Arrow caused it to be changed to this present form.

[63] See, for example, Guilbaud [133], May [194, 195, 196], Fleming [110], Weldon [309], Gorman [128], Hildreth [142], Inada [147, 148], Harsanyi [138], Vickrey [301], Murakami [208, 209, 210], Dummett and Farquharson [79], Tullock [291], Theil [285], Nicholson [213] and Sen [254, 255].

choice from any given environment is an aggregation of individual preferences", (p. 103).

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